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NON-IMPROVED UNIFORM TAIL ESTIMATES FOR NORMED SUMS OF INDEPENDENT RANDOM VARIABLES WITH HEAVY TAILS, with applications.

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Abstract.

We obtain an uniform tail estimates for natural normed sums of independent random variables (r.v.) with regular varying tails of distributions.

We give also many examples on order to show the exactness of offered estimates and discuss some applications in the method Monte-Carlo and statistics, and obtain the sufficient conditions for Central and stable limit theorem in the Banach space of continuous function.

There are considered a slight generalization on a random variables with super-heavy tails and martingale difference scheme.

Key words and phrases: Tail function, normed, centered and ordinary random variables (r.v.), continuous, regular and slowly varying functions, moments and moment spaces, characteristical functions, Central and Stable Limit Theorem (CLT, SLT), Monte-Carlo method, statistics, Gaussian and stable distribution, Orlicz and Grand Lebesgue spaces of random variables, accomplishing infinite divisible distribution, martingale and martingale differences, Banach space.

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1 Introduction. Notations. Statement of problem.

Let $\xi = \xi_1$ be a random variable defined on some sufficiently rich probabilistic space $(\Omega, \mathcal{A}, \mathbf{P})$ with regularly varying as $x \to \infty$ tail behavior:

$$T(x) = T_\xi(x) = x^{-r} \ \log^\gamma(x) \ L(\log x), r = \text{const} > 0, \ x > e, \tag{1.1}$$

where as ordinary the tail function $T(x) = T_\xi(x)$ for the r.v. ξ may be defined as follows:

$$T(x) = T_\xi(x) = \mathbf{P}(|\xi| \ge x), \ x > 0; \tag{1.2}$$

and $L = L(x)$ is positive continuous slowly varying as $x \to \infty$ function.

Further, we denote as $\phi(t) = \phi_\xi(t)$ the characteristical function of the r.v. ξ :

$$\phi(t) = \phi_\xi(t) = \mathbf{E}\exp(it\xi) \tag{1.3}$$

and by $\psi(t) = \psi_\xi(t)$ its addition:

$$\psi(t) = \psi_\xi(t) = 1 - \mathbf{Re}[\phi_\xi(t)] = \mathbf{E}(1 - \mathbf{Re}[\exp(it\xi)]). \tag{1.4}$$

Note that if the r.v. ξ has symmetrical distribution, then

$$\psi(t) = \psi_\xi(t) = 1 - \phi_\xi(t) = \mathbf{E}(1 - \exp(it\xi)). \tag{1.4a}$$

Denote also in symmetrical case

$$\overline{\psi}(t) = \overline{\psi}_\xi(t) = \sup_{\lambda \in (0,1)} \left[\frac{\psi(\lambda t)}{\psi(\lambda)} \right];$$

$$K(p) = 2\pi^{-1}\Gamma(1+p)\sin(\pi p/2), \ p \in (0,2).$$

Note that the function

$$d(t,s) = d_\xi(t,s) = \sqrt{\psi(t-s)}$$

is a bounded translation invariant continuous *distance* between a two points on the real line R.

Let $\xi(k), \ k = 1, 2, \ldots, n$ be independent copies of ξ; we define the *natural norming* sequence $\{b(n)\}$ as follows: $b(1) = 1$ and for $n = 2, 3, \ldots$ as a positive solution of an equation:

$$\psi_\xi(1/b(n)) = n. \tag{1.5}$$

M.Braverman in [5] proved that the non-random sequence $\{b(n)\}$ is natural norming sequence for the sum of independent variables $\{\xi(i)\}$.

Note that for sufficiently greatest values n the values $\{b(n)\}$ exists, is unique and $\lim b(n) = \infty, \ n \to \infty$.

For instance, if $\mathbf{Var}(\xi) \in (0, \infty)$, then $b(n) \asymp \sqrt{n}$, (the classical norming sequence.)

Another example. Assume that the r.v. ξ has a symmetric stable distribution of order r :

$$\text{Law}(\xi) = St(r) \ \Leftrightarrow \phi_\xi(t) = \exp\left(-|t|^r\right), \ r \in (0,2).$$

In this case $b(n) \asymp \ n^{1/r}$.

Let us denote

$$S(n) = b(n)^{-1} \sum_{k=1}^n \xi(k), \tag{1.6}$$

$$U_\xi(x) = U(x) = \sup_{n=1,2,\ldots} T_{S(n)}(x). \tag{1.7}$$

The function $U(x)$ is *uniform tail function* for natural normed sums of independent random variables $\xi(k), \ k = 1, 2, \ldots, n.$

Our aim is estimate of the uniform tail function $U(x)$ **through the source tail function** $T(x)$.

We will distinguish and investigate all the cases $r \in (0, 2)$ (heavy tails), $r = 2$ (intermediate tails), and $r > 2$ (moderate tails); $\mathbf{E}|\xi|^r = \infty$ (infiniteness of main moment) and $\mathbf{E}|\xi|^r < \infty$ (finiteness of main moment).

In the last section we consider the case of super-heavy tails, i.e. when the tail function decreases logarithmicaly.

For the *exponential* tail function

$$T(x) = \exp\left(-Cx^m\right), \ m = \text{const} \in [1, \infty)$$

the non-improved estimate of the function $U(x)$ has a view:

$$U(x) \le \exp\left(-C_1(C, m)x^{\min(m,2)}\right),$$

see [4], [25], [33].

Notice that in exponential case the critical value of the parameter m is also the value $m = 2$.

The *moment* estimates for $S(n)$ in the case when $r \le 2$ and $\mathbf{E}|\xi|^r < \infty$ is investigated in [5]. Another estimates (tail and moments) see in [1], [9], [17], [21], [27], [30], [39], [47] etc.

The applications of these estimates, e.g. in the Monte-Carlo method for errors estimates for integrals with infinite dispersion see in [11], [33]. In detail, let us consider the problem of numerical computation of an absolute convergent integral (multiple, in general case) of a view:

$$I = \int_D f(y) \ \nu(dy),$$

where $\nu(\cdot)$ is probabilistic measure on the set $D: \ \nu(D) = 1.$

Let $\tau(k), \ k = 1, 2, \ldots, n$ be independent r.v. with distribution ν : $\mathbf{P}(\tau(k) \in A) = \nu(A)$. The Monte-Carlo estimation I_n of an integral I is

$$I_n = n^{-1} \sum_{k=1}^n f(\tau(k)).$$

Suppose for some $r \in (1, 2)$

$$\mathbf{E}|f(\tau(1))|^r < \infty$$

or more generally that the r.v. $f(\tau(k)) - I$ satisfies the condition (1.1); note that in the case $r > 2$ it may be used for error evaluating the classical Central Limit Theorem.

In order to construct a non-asymptotical confidence interval for I of a reliability $1 - \delta, \ \delta = 0.05; 0.01$ etc. we consider the probability

$$U_n(x) = \mathbf{P}\left(b(n)^{-1}\left|\sum_{k=1}^n (f(\tau(k) - I))\right| > x\right).$$

Note that $U_n(x) \le U(x)$; therefore, we conclude denoting by $X(\delta)$ the solution of an equation

$$U(X(\delta)) = \delta$$

that with probability at least $1 - \delta$

$$|I_n - I| \le X(\delta) b(n)/n.$$

Obviously, under condition (1.1)

$$\lim_{n\to\infty} b(n)/n = 0.$$

Analogous application appears in statistics. Indeed, let us consider the following classical scheme of date elaboration.

$$\eta(k) = \theta + \xi(k), \ \ k = 1, 2, \ldots, n;$$

where θ is unknown deterministic parameter, $\{\xi(k)\}$ are i.,i.d. centered r.v. satisfying the condition (1.1) with $r > 1$ (additive noise with heavy tail).

The consistent estimation of the parameter θ has a view

$$\hat{\theta}_n = n^{-1} \sum_{k=1}^{n} \eta(k).$$

We conclude as before that with probability at least $1 - \delta$ under formulated above conditions and notations

$$\left| \hat{\theta}_n - \theta \right| \le X(\delta) b(n)/n.$$

Remark 1.1. Suppose

$$T(x) = 0 \left(x^{-r} \right), \quad x \to \infty, \ r = \text{const} \in (0, 2),$$

then the distribution of $\xi(k)$ dominated by symmetric stable distribution $C \cdot St(r)$; then by virtue of a main result of S.Kwapien [27] $b(n) \sim n^{-1/r}$ and

$$U(x) = 0 \left(x^{-r} \right), \ r = \text{const} \in (0, 2).$$

Remark 1.2. Suppose

$$\mathbf{E}|\xi|^r < \infty, \ r = \text{const} \in (0, 2).$$

This condition is equivalent to the follows:

$$\int_e^\infty x^{-1} \ \log^\gamma(x) \ L(\log x) \ dx < \infty.$$

We conclude using famous result belonging to B. von Bahr and C.-G. Esseen [1] that again $b(n) \sim n^{-1/r}$ and

$$\sup_n \mathbf{E}|S(n)|^r \le 2\mathbf{E}|\xi|^r,$$

therefore

$$U(x) = o\left(x^{-r}\right), \ x \to \infty, \ r = \text{const} \in (0,2).$$

The paper is organized as follows. In the next section we consider the so-called case of heavy tails: $r \in (0,2)$. Third section is devoted mainly to the consideration of intermediate case $r = 2$. Fourth section contains the investigation of the case moderate tails: $r \in (2, \infty)$.

In the fifth section we generalize preceding results on the martingale case, i.e. when the summands $\{\xi(i)\}$ are centered martingale differences relative some filtration $\{F(i)\}$. The sixth section contains the tail evaluating for normed sums of the r.v. with superheavy, i.e. logarithmical, tails of distribution. In the 7^{th} section we obtain sufficient conditions for Stable and Central Limit Theorems for heavy tail random fields in the Banach space of continuous functions on the compact metric spaces.

The last section is devoted to concluding remarks.

The letter C, with or without subscript, denotes a finite positive non essential constants, not necessarily the same at each appearance.

2 Main result: heavy tails

Theorem 2.1. Let the r.v. ξ has a symmetrical distribution. Then

$$U(x) \le 0.5x \int_{-2/x}^{2/x} \overline{\psi}(t) \ dt, \ x > 0. \tag{2.1}$$

Proof. We will use the well-known inequality, which is true for arbitrary r.v. η :

$$T_\eta(2/a) \le a^{-1} \int_{-a}^{a} \psi_\eta(t) \ dt, \ a = \text{const} > 0. \tag{2.2}$$

Here $a = 2/x$. We denote

$$V(n) = \sum_{k=1}^{n} \xi(k) = b(n) S(n).$$

M.Braverman in [5] proved that

$$\psi_{V(n)}(t) \le \sum_{k=1}^{n} \psi_{\xi(k)}(t) = n \ \psi_\xi(t) = n \ \psi(t),$$

therefore

$$\psi_{S(n)}(t) \le n \ \psi(t/b(n)).$$

We get:

$$\psi_{S(n)}(t) \le \sup_n [n \ \psi(t/b(n))] \le \sup_n [n \ \psi(t \cdot \psi^{-1}(1/n))] \le$$

$$\sup_{\lambda \in (0,1)} \frac{\psi(\lambda t)}{\psi(\lambda)} = \overline{\psi}(t). \tag{2.3}$$

It remains to use the inequality (2.2).

Theorem 2.2. Suppose again the r.v. ξ has a symmetrical distribution and let $r \in (0,2)$. Then

$$U(x) \le \inf_{p \in (0,r)} \left[K(p)\ x^{-p} \int_0^\infty \overline{\psi}(t)\ t^{-p-1}\ dt \right]. \tag{2.4}$$

Proof. We will use the famous formula:

$$\mathbf{E}|\eta|^p = K(p) \int_0^\infty \psi_\eta(t)\ t^{-p-1}\ dt, \tag{2.5}$$

see [5], [24]. Here $\eta = S(n)$ and we may replace in the last inequality $\overline{\psi}_{S(n)}(t)$ instead $\psi_\eta(t)$.

Estimating $\overline{\psi}_{S(n)}(t)$ by means of inequality (2.3), we obtain the assertion of theorem 2.2.

Corollary 2.1. Choosing in (2.4) the value $p = r - C/\log x,\ x > e^2$, we conclude:

$$U(x) \le e^C \left[K(r - C/\log x)\ x^{-r} \int_0^\infty \overline{\psi}(t)\ t^{-r+C/\log x - 1}\ dt \right]. \tag{2.6}$$

If for example

$$\overline{\psi}(t) \le |t|^r\ |\log t|^\beta,\ |t| < 1/e,\ \beta = \text{const} \ge 0, \tag{2.7}$$

then the optimal value of the constant C in (2.5) is $C = \beta + 1$ and hence for the values $x > \exp(2(\beta+1)/r)$

$$U(x) \le e^{\beta+1}\ (\beta+1)^{-\beta-1}\ K(r - (\beta+1)/\log x)\ x^{-r}\ [\log x]^{\beta+1} \le$$

$$C_1(\beta, r)\ x^{-r}\ [\log x]^{\beta+1}. \tag{2.8}$$

We investigate further in this section the case only when in the condition (1.1) $r \in (0,2)$.

Definition 2.1. We will say that the r.v. η has a *regular tail,* write: $\mathrm{Law}(\eta) \in RT$, if is true the inverse inequality to the inequality (2.2) up to multiplicative constant, i.e. when there exists constant $C_1 > 0$ so that for all sufficiently small positive values a

$$T_\eta(2/a) \ge C_1 a^{-1} \int_{-a}^a \psi_\eta(t)\ dt. \tag{2.9}$$

For instance, if the non-trivial r.v. ξ satisfies the condition (1.1) with $r \in (0,2]$, then $\mathrm{Law}(\xi) \in RT$.

Conversely, if $\xi \neq 0$ and if for some $\Delta = \text{const} > 0$ $\mathbf{E}|\xi|^{2+\Delta} < \infty$, then $\text{Law}(\xi) \notin RT$. Namely, it follows from Tchebychev's inequality

$$T_\xi(2/a) \le C_2 a^{2+\Delta}, \ a \in (0,1),$$

but

$$a^{-1} \int_{-a}^{a} \psi_\xi(t) \ dt \ge C_3 a^2.$$

Definitions 2.2. We will say that the r.v. ξ (more exactly, the distribution $\text{Law}(\xi)$ of the r.v. ξ) belongs to the class MI (Monotonically Increasing), write: $\text{Law}(\xi) \in MI$, if the function $\lambda \to \theta(\lambda), \ \lambda \in (0,1)$, where

$$\theta(\lambda) = \frac{\psi(\lambda t)}{\psi(\lambda)}, \tag{2.10}$$

monotonically non-decreased for all the values t in some neighborhood $t \in [0, \Delta), \ \Delta > 0$.

Analogously, we will say that the r.v. ξ (more exactly, the distribution $\text{Law}(\xi)$ of the r.v. ξ) belongs to the class MD (Monotonically Decreasing), write: $\text{Law}(\xi) \in MD$, if the function $\lambda \to \theta(\lambda), \ \lambda \in (0,1)$, where

$$\theta(\lambda) = \frac{\psi(\lambda t)}{\psi(\lambda)}, \tag{2.11}$$

monotonically non-increased for all the values t in some neighborhood $t \in [0, \Delta), \ \Delta > 0$.

Obviously, if $\text{Law}(\xi) \in MD$, then

$$\overline{\psi}(t) = \psi(t), \ t \in [0, \Delta] \tag{2.12}$$

and if $\text{Law}(\xi) \in MI$ and if the distribution of the r.v. ξ satisfies the condition (1.1), then

$$\overline{\psi}(t) = t^r, \ t \in [0, \Delta]. \tag{2.13}$$

For instance, if the distribution of the r.v. ξ satisfies the condition (1.1) and $\gamma > 0$, then $\text{Law}(\xi) \in MD$ and $\overline{\psi}(t) = \psi(t), \ t \in [0, \Delta]$; if the distribution of the r.v. ξ satisfies the condition (1.1) and $\gamma < 0$, then $\text{Law}(\xi) \in MI$ and $\overline{\psi}(t) = t^r, \ t \in [0, \Delta]$.

Consider now the case $\gamma = 0$. Suppose

$$T(x) = T_\xi(x) = x^{-r} \ [\log\log x]^\kappa \ L(\log\log x), \kappa = \text{const}, \ x > e^3, \tag{2.14}$$

where as before $L = L(z)$ is slowly varying as $z \to \infty$ positive function.

If in (2.13) $\kappa > 0$, then $\text{Law}(\xi) \in MD$ and if in (2.13) $\kappa < 0$, then $\text{Law}(\xi) \in MI$.

Theorem 2.3. If $\text{Law}(\xi) \in RT \cap MD$, then we propose the following non-improvable up to multiplicative constant estimates:

$$T(x) \le U(x) \le C(r,L) T(x), \ x > x_0 = \text{const} > 1. \tag{2.15}$$

Proof. The left inequality in (2.14) is trivial; it remains to prove the right-hand inequality.

We will use the following fact: if the equality (1.1) holds and $r \in (0,2)$, then as $t \to 0+$

$$\psi_\xi(t) \sim C_3(r) \ t^r \ |\log t|^\gamma \ L(|\log t|), \tag{2.16}$$

where

$$C_3(r) = \Gamma(1-r) \ \cos(\pi r/2), \ r \ne 1,$$

[19], p. 86-87, see also [5].

As long as $\text{Law}(\xi) \in MD$,

$$\overline{\psi}_\xi(t) \sim \psi_\xi(t), \ t \in (0, \Delta). \tag{2.17}$$

Since $0 < r < 2$, we conclude by virtue of the equality (2.16) and the condition $\text{Law}(\xi) \in RT$ that

$$U(x) \le C(r,L) x^{-r} \ [\log x]^\gamma \ L(\log x) = C(r,L) T(x), \tag{2.18}$$

Q.E.D.

Analogously may be proved the following result.

Theorem 2.4. If $\text{Law}(\xi) \in RT \cap MI$, then

$$U(x) \le C_1(r,L) x^{-r}, \ x > x_0 = \text{const} > 1, \tag{2.19}$$

and the last inequality is exact, for example, for the symmetric stable distribution ξ.

Notice that the member $L(\cdot)$ is absent in the right-hand of the inequality (2.19). This expression included in the definition on the norming sequence $\{b(n)\}$.

Examples.

A. Suppose the r.v. ξ is symmetrically distributed, satisfies the condition (1.1) and $\gamma > 0$; then

$$U(x) \le C(r,\gamma,L) x^{-r} \ [\log x]^\gamma \ L(\log x) = C(r,L) T(x), \ x > e^2.$$

B. Suppose now the r.v. ξ is symmetrically distributed, satisfies the condition (1.1) and $\gamma < 0$; then

$$U(x) \le C_1(r,L) x^{-r}, \ x > x_0 = \text{const} > 1.$$

C. Consider now up to end of this section the case when $\gamma = 0$. Suppose

$$T(x) = T_\xi(x) = x^{-r} \ [\log \log x]^\kappa \ L(\log \log x), \kappa = \text{const}, \ x > e^3,$$

where as before $L = L(z)$ is slowly varying as $z \to \infty$ positive function.

If $\kappa > 0$, then

$$U(x) \le C(r, \kappa, L) x^{-r} \ [\log \log x]^\kappa \ L(\log \log x), \ x > e^3.$$

D. If $\kappa < 0$, we conclude

$$U(x) \le C_1(r, \kappa, L) x^{-r}, \ x > 1.$$

Remark 2.1. Note that the application of theorem 2.2. give us more slight result, namely

$$U(x) \le C \ x^{-r} \ [\log x]^{\gamma+1} \ L(\log x) = T(x), \ x > e.$$

3 Main result: intermediate case.

We consider in this section the case when in the condition (1.1) $r = 2$.

Theorem 3.1. Suppose the r.v. ξ is symmetrically distributed, satisfies the condition (1.1) for $r = 2$ and for some $\gamma = \text{const} \in R$; then:

A. If $\gamma \ge -1$, then

$$U(x) \le C(L) x^{-2} \ [\log x]^{\gamma+1} \ L(\log x) = C(r, \gamma, L) \ T(x) \ \log x, \ x > e; \quad (3.1a)$$

B. If $\gamma < -1$, then

$$U(x) \le C(L) x^{-2}, \ x > 1. \quad (3.1b)$$

Proof. Let us introduce a following function:

$$H(x) = -\int_0^x u^2 \ dT_\xi(u); \quad (3.2)$$

then

$$\psi_\xi(t) \sim 0.5 \ t^2 \ H(1/|t|), \ t \to 0,$$

see [19], p.86-88; see also [5].

In the considered case

$$\psi_\xi(t) \sim C_4(\gamma, L) \ t^2 \ |\log t|^{\gamma+1} \ L(\log x), \ t \to 0. \quad (3.3)$$

The remained part of the proof is at the same as in theorems 2.3-2.4.

Remark 3.1. Note concerning the lower bound for $U(x)$ in considered case $r = 2$ that we can show only the trivial bound

$$U(x) \ge x^{-2} \ [\log x]^\gamma \ L(\log x) = T(x), \ x > e; \quad (3.4)$$

Note that there is a "gap" of a view "degree of a logarithmic term" $[\log x]^{\Delta}, \ \Delta > 0$ between upper and lower bound for the uniform tail of probability $U(x)$.

4 Main result: moderate tails

We concentrate our attention in this section on the case when in the equality (1.1) $r > 2$, *and suppose* $\mathbf{E}\xi = 0$.

Recall that in this case the norming sequence $b(n)$ is ordinary: $b(n) = \sqrt{n}$.

In order to formulate and prove the main result in this case, we recall here for reader convenience some facts about so-called Grand Lebesgue Spaces [13], [14], [20], [29] etc. or equally "moment" spaces of random variables defined on fixed probabilistic space $(\Omega, \mathcal{A}, \mathbf{P})$; more detail description see in [25], [29], [33], [34].

Let us consider the following norm (the so-called "moment norm") on the set of r.v. defined in our probability space by the following way: the space $G(\nu) = G(\nu; r)$ consist, by definition, on all the r.v. with finite norm

$$||\xi||G(\nu) \stackrel{def}{=} \sup_{p \in (2,r)} [|\xi|_p / \nu(p)], \ |\xi|_p := \mathbf{E}^{1/p} |\xi|^p. \tag{4.1}$$

Here $r = \mathrm{const} > 2, \ \nu(\cdot)$ is some continuous positive on the *semi-open* interval $[1, r)$ function such that

$$\inf_{p \in (2,r)} \nu(p) > 0, \ \nu(p) = \infty, \ p > r;$$

and as usually

$$|\xi|_p \stackrel{def}{=} [\mathbf{E}|\xi|^p]^{1/p}$$

We will denote

$$\mathrm{supp}(\nu) \stackrel{def}{=} [1, r) = \{p : \nu(p) < \infty\}.$$

The case $r = +\infty$ is investigated in [25], [33], [34]; therefore, we suppose further $2 < r < \infty$.

Let ξ be a r.v. such that

$$p > r \Rightarrow |\xi|_p \stackrel{def}{=} [\mathbf{E}|\xi|^p]^{1/p} = \infty$$

The *natural* function $\nu_\xi(p)$ may be defined as follows:

$$\nu_\xi(p) := |\xi|_p = [\mathbf{E}|\xi|^p]^{1/p}. \tag{4.2}$$

Obviously,

$$||\xi||G(\nu_\xi) = 1.$$

The *natural* function for the *family* $\{\xi(\cdot)\}$ of a r.v. $\{\nu(\cdot)\} = \{\nu_{\xi(k)}(p)\}, \ k = 1, 2, \ldots$ may be defined as follows:

$$\nu_{\{\xi\}}(p) := |\xi|_p = \sup_k \left[\mathbf{E}|\xi(k)|^p\right]^{1/p}, \tag{4.2a}$$

if there exists and is finite.

The complete description of a possible natural functions see in [34], [33], chapter 1, section 3.

Example 4.1. Suppose the r.v. ξ satisfies the condition (1.1) for $r > 1$ and $\gamma > -1$. Then (see [29], [34]) for the values $p \in [1, r), \ p \to r - 0$

$$\mathbf{E}|\xi|^p \sim C(r, \gamma, L) \ (r - p)^{-\gamma - 1} \ L(1/(r - p)).$$

Note that an inequality $\psi(r - 0) < \infty$ is equivalent to the moment restriction $|\xi|_r < \infty$.

We recall now the relations between moments for r.v. $\xi, \ \xi \in G(\nu, r)$ and its tail behavior. Namely, for $p < r$

$$|\xi|_p = \left[p \int_0^\infty u^{p-1} \ T_{|\xi|}(u) \ du\right]^{1/p},$$

therefore

$$||\xi||G(\nu; r) = \sup_{p < r} \left\{\left[p \int_0^\infty u^{p-1} \ T_{|\xi|}(u) \ du\right]^{1/p} / \nu(p)\right\}.$$

Conversely, if the r.v. ξ belongs to the space $G(\nu, r)$, then

$$T_{|\xi|}(x) \le T^{(||\xi||G(\nu; r) \cdot \nu)}(x),$$

where by definition

$$T^{(\nu)}(x) \stackrel{def}{=} \inf_{p \in (1, r)} \left[\nu^p(p) \ / x^p\right], \ x > 0.$$

Example 4.2. If

$$\mathbf{E}|\xi|^p \le C_2 \ (r - p)^{-\gamma - 1} \ L(1/(r - p)), \ p < r,$$

then

$$T_{|\xi|}(x) \le C_4(r, \gamma, L) \ x^{-r} \ [\log x]^{\gamma + 1} \ L(\log x), \ x > e.$$

Notice that there is a "logarithmic gap" between upper and lower tail and moment relations. This gap is essential, see [29], [34]. Let us consider the following

Example 4.3. Let ζ be a discrete r.v. with distribution

$$\mathbf{P}\left(\zeta = \exp(e^k)\right) = C_5 \ \exp\left(\beta r k - r e^k\right),$$
$$k = 1, 2, \ldots, \ \beta = \text{const} > 0,$$

where obviously

$$1/C_5 = \sum_{k=1}^{\infty} \exp\left(\beta r k - r e^k\right).$$

We conclude after some calculations:

$$|\zeta|_p \asymp (r-p)^{-\beta},$$

but for the sequence $x(k) = \exp(\exp(k))$

$$T_\zeta(x(k)) \ge c_6 [\log x(k)]^\beta \ (x(k))^{-r}.$$

Let us introduce the so-called Rosenthal's constants (more exactly, Rosental's functions) [41] $R(p)$ as follows:

$$R(p) = \sup_{\zeta(k):\mathbf{E}\zeta(k)=0} \frac{|\sum_{k=1}^n \zeta(k)|_p}{\max(|\sum_{k=1}^n \zeta(k)|_2, [\sum_{k=1}^n |\zeta(k)|_p^p]^{1/p})}, \tag{4.3}$$

where the supremum in (4.3) is calculated over all the sequences of independent centered random variables $\{\zeta(k)\}$ with condition $|\zeta(k)|_p < \infty$.

This constants was intensively investigated in many publications, see,e.g. [41], [9], [17], [21], [22], [28], [35], [42], [46] etc. It is known, for instance, that there exists an absolute constant C_R, which is exactly calculated in [35]: $C_R \approx 1.77638$, so that

$$R(p) \le C_R \frac{p}{e \cdot \log p}, \ p \in [2, \infty). \tag{4.4}$$

Note that for symmetrical distributed r.v. $\{\zeta(k)\}$ the correspondent Rosenthal's constant C_R is (approximately) equal to ≈ 1.53573.

It follows from inequality (4.4) that if the r.v. $\{\eta(k)\}, \ k = 1, 2, \ldots, n$ are centered, i., i.d., and $|\eta(1)|_p < \infty, \ p \ge 2$, then

$$\sup_n \left| n^{-1/2} \sum_{k=1}^n \eta(k) \right|_p \le R(p) \ |\eta(1)|_p. \tag{4.5}$$

Suppose now that the *centered* r.v. ξ belongs to some space $G(\nu; r), \ r \in (2, \infty)$; this condition is satisfied, if ξ satisfied the condition (1.1) for some $r \in (2, \infty)$ and $\mathbf{E}\xi = 0$.

For instance, the function $\nu = \nu(p)$ may be the natural function for the r.v. $\xi : \nu = \nu_\xi(p)$.

Theorem 4.1. We have the following non-improved inequality in the terms of $G(\nu; r)$ norms:

$$\sup_n ||n^{-1/2} \sum_{k=1}^n \xi(k)||G(\nu; r) \le C(\nu; r)||\xi||G(\nu; r); \tag{4.6a}$$

$$||\xi||G(\nu_\xi) \le \sup_n ||n^{-1/2} \sum_{k=1}^n \xi(k)||G(\nu_\xi) \le C(\nu_\xi)||\xi||G(\nu_\xi). \tag{4.6b}$$

Proof. Let $\xi \in G(\nu;r)$ and $\mathbf{E}\xi = 0$. We can and will assume without loss of generality $||\xi||G(\nu;r) = 1$. It follows from this equality

$$|\xi|_p \le \nu(p),\ \ p \in [2,r).$$

We conclude by means of inequality (4.5) by virtue of inequality $p < r$:

$$\sup_n \left| n^{-1/2} \sum_{k=1}^n \xi(k) \right|_p \le R(p)|\xi|_p \le R(p)\ \nu(p) \le C_4(r)\nu(p), \tag{4.7}$$

$$\sup_n \sup_{p \in [2,r)} \left[\left| n^{-1/2} \sum_{k=1}^n \xi(k) \right|_p / \nu(p) \right] \le C_4(r),$$

which is equivalent to the assertion (4.6a) of our theorem.

The proposition (4.6b) follows from (4.6a) after choosing $\nu(p) = \nu_\xi(p)$.

As a consequence:

Theorem 4.2.

$$U(x) \le C(r, \mathrm{Law}(\xi))\ T^{(\nu)}(x). \tag{4.8}$$

Example 4.4. Let $\mathbf{E}\xi = 0$ and suppose the r.v. ξ satisfies the condition (1.1). Then

$$x^{-r}\ \log^\gamma(x)\ L(\log x) \le U(x) \le C_6(r,\gamma,L)\ x^{-r}\ \log^{\gamma+1}(x)\ L(\log x),\ \ x > e. \tag{4.9}$$

Remark 4.1. *We can not prove that the power of the logarithmic multiplier* $\gamma + 1$ *in the right-hand side of last inequality is not improvable.*

But we can offer the following example.

Example 4.5. Let ζ be the r.v. from the example 4.3., put $\zeta^o = \zeta - \mathbf{E}\zeta$.

Define a r.v. θ as a r.v. with *accomplishing* infinite divisible distribution to the distribution ζ^o. This imply that

$$\theta = \sum_{m=1}^{\tau} \zeta^o(m),$$

where $\zeta^o(m)$ are independent copies of ζ^o, the r.v. τ gas a standard Poisson distribution with unit parameter: $\mathbf{P}(\tau = j) = 1/(e\, j!), j = 0,1,2,\ldots;\ \sum_{m=1}^0 = 0$.

Obviously,

$$\phi_\theta(t) = \exp\left[\phi_{\zeta^o}(t) - 1\right].$$

It follows from the last formula that the asymptotical behavior of the function $\phi_\theta(t)$ as $t \to 0$ or equally the tail behavior $T_\theta(x)$ as $x \to \infty$ alike the asymptotical behavior of the function $\phi_{\zeta^o}(t),\ t \to 0$ and correspondingly the behavior $T_{\zeta^o}(x)$ as $x \to \infty$. For instance, Yu.V.Prohorov in [39] proved the following inequality:

$$\mathbf{P}\left(\sum_{k=1}^{n}\theta(k) > x\right) \le 8\ \mathbf{P}\left(\sum_{k=1}^{n}\zeta^{o}(k) > x/2\right). \tag{4.10}$$

Let us denote

$$\Theta_n = n^{-1/2}\sum_{k=1}^{n}\theta(k),$$

where $\{\theta(k)\}$ are independent copies of the r.v. θ. We conclude:

$$|\theta|_p \asymp (r-p)^{-\beta},\ p \in [1, r),$$

but there exists a constant $q \in (0,1)$ such that for the sequence $x(k) = \exp(\exp(k))$ and for all the values $n = 1, 2, \ldots$

$$T_{\Theta_n}(x(k)) \ge C_7\ q^n\ [\log x(k)]^{\beta}\ (x(k))^{-r}.$$

Remark 4.2. It is not necessary to suppose in this section that the r.v. $\xi(i)$ are identical distributed; it is sufficient to assume that the r.v. $\{\xi(i)\}$ are independent,centered and such that

$$\sup_i T_{\xi(i)}(x) \le x^{-r}\ \log^{\gamma}(x)\ L(\log x), r = \text{const} > 2,\ x > e. \tag{4.11}$$

Remark 4.3. The application of the interpolation technique, see for instance [23], gives us more slight result as in the theorems 4.1-4.2. For example, we conclude under conditions in example 4.4

$$U(x) \le C_7(r,\gamma,L)\ x^{-r}\ \log^{\gamma+1}(x)\ \log\log x\ L(\log x),\ x > e^{e}. \tag{4.12}$$

We give briefly here the proof of last inequality, as long as it has by our opinion self-contained interest.

Let the centered r.v. $\xi(k),\ k = 1, 2, n$ be i., i.d. and satisfy the equality (1.1). Let also $\delta = \text{const} > 0$. Consider the following Orlicz's function:

$$N(u) = N_{r,\gamma,L}(u) = \delta^{-1}\ |u|^r\ \left(\log^{-\gamma-\delta}|u|\right)\ L(\log|u|),\ u > e, \tag{4.13}$$

and as usually $N(u) = C\ u^2, |u| \le e$, so that the function $N = N(u)$ is continuous.

We can realize the sequence $\xi(k),\ k = 1, 2, \ldots$ on the direct (Cartesian) product of probabilistic spaces $(\Omega_k, \mathcal{A}_k, \mathbf{P_k})$:

$$(\Omega, \mathcal{A}, \mathbf{P}) = \otimes_{k=1}^{\infty}(\Omega_k, \mathcal{A}_k, \mathbf{P_k})$$

so that $\xi(k)$ is symmetrical distributed measurable function $\xi(k) : \Omega_k \to R$.

As long as $r > 2$, there is two numbers p_1, p_2 for which $2 < p_1 < r < p_2 < \infty$, for example, $p_1 = 0.5(2+r),\ p_2 = r+1$.

Define the following linear operator acting only on the centered random vectors:

$$T(\xi(1), \xi(2), \ldots, \xi(n)) = n^{-1/2} \sum_{k=1}^{n} \xi(k). \tag{4.14}$$

From the Rosental's inequality follows that T is bounded operator from the spaces $L_{p_j}, j = 1, 2$ into at the same spaces. We conclude by virtue of interpolation theorem belonging to A.Yu.Karlowich and L.Maligranda [23] that the operator T is bounded uniformly in n operator from the Orlicz's space $M(N_{r,\gamma,L}, \Omega, \mathbf{P}) \stackrel{def}{=} M(r, \gamma, L)$ into at the same space:

$$\sup_n ||S(n)||M(r, \gamma, L) \le C(r, \gamma, L) \ \sup_k ||\xi(k)||M(r, \gamma, L). \tag{4.15}$$

It is easy to calculate that the right-hand side of last inequality is uniformly in k and $\delta \in (0, 0.08)$ bounded and therefore

$$\sup_{\delta \in (0,0.08)} \ \sup_n ||S(n)||M(r, \gamma, L) \le C_2(r, \gamma, L) < \infty. \tag{4.16}$$

Since the function $N(u) = N_{r,\gamma,L}(u)$ satisfies the Δ_2 condition, we get: (see [26], chapter 2, section 9)

Proposition 4.1.

$$\sup_{\delta \in (0,0.08)} \ \sup_n \mathbf{E} N_{r,\gamma,L}(S(n)) \le C_2(r, \gamma, L) \sup_{\delta \in (0,0.08)} \ \sup_k \mathbf{E} N_{r,\gamma,L}(\xi(k)). \tag{4.17}$$

Tchebychev's inequality give us:

$$\sup_{\delta \in (0,0.08)} \ \sup_n T_{S(n)}(x) \le C_3(r, \gamma, L) \ x^{-r} \ \delta^{-1} \ \log^{\gamma+1+\delta} x \ L(\log x).$$

We obtain choosing $\delta = (\log \log x)^{-1}$

$$U(x) \le C_7(r, \gamma, L) \ x^{-r} \ \log^{\gamma+1}(x) \ \log \log x \ L(\log x), \ \ x > e^e,$$

Q.E.D.

5 Martingale generalization

In this fifth section we generalize preceding results on the martingale case, i.e. when the summands $\{\xi(i)\}$ are centered $\mathbf{E}\xi(i) = 0$ martingale differences relative some filtration $\{F(i)\}: \ \ F(0) = \{\emptyset, \ \Omega\}, F(i) \subset F(i+1) \subset \mathcal{A}$. This imply, by definition,

$$\mathbf{E}\xi(k)/F(k-1) = 0, \ \ k = 1, 2, \ldots.$$

We will use in the sequel the following generalization of Rosenthal's inequality for centered martingales, see [36]:

$$\left| n^{-1/2} \sum_{k=1}^{n} \xi(k) \right|_p \le p \sqrt{2} \sup_i |\xi(i)|_p, \ p \ge 1. \tag{5.1}$$

Theorem 5.1. We have alike in the fourth section in the case $2 < r < \infty$ the following non-improved inequality in the terms of $G(\nu; r)$ norms:

$$\sup_n ||n^{-1/2} \sum_{k=1}^{n} \xi(k)||G(\nu; r) \le C(\nu; r)||\xi||G(\nu; r); \tag{5.2}$$

$$\sup_i ||\xi(i)||G\left(\nu_{\{\xi\}}\right) \le \sup_n ||n^{-1/2} \sum_{k=1}^{n} \xi(k)||G\left(\nu_{\{\xi\}}\right) \le$$

$$C(\nu_{\xi}) \sup_i ||\xi(i)||G\left(\nu_{\{\xi\}}\right). \tag{5.3}$$

Recall that

$$\nu_{\{\xi\}}(p) \stackrel{def}{=} \sup_i ||\xi(i)||G(\nu; r).$$

Proof. Let $\sup_i ||\xi(i)||G(\nu; r) < \infty$ and $\mathbf{E}\xi = 0$. We can and will assume without loss of generality $\sup_i ||\xi(i)||G(\nu; r) = 1$. It follows from this equality

$$\sup_i |\xi(i)|_p \le \nu(p), \ p \in [2, r).$$

We conclude by means of inequality (5.1) by virtue of inequality $p < r$:

$$\sup_n |n^{-1/2} \sum_{k=1}^{n} \xi(k)|_p \le p \sqrt{2} \ |\xi|_p \le p \sqrt{2} \ \nu(p) \le C_4(r)\nu(p), \tag{5.4}$$

$$\sup_n \sup_{p \in [2,r)} \left[\left| n^{-1/2} \sum_{k=1}^{n} \xi(k) \right|_p / \nu(p) \right] \le C_4(r),$$

which is equivalent to the assertion of our theorem.

The second proposition of one follows from (5.4) after choosing $\nu(p) = \sup_i \nu_{\xi(i)}(p)$.

As a consequence:

Theorem 5.2.

$$U(x) \le C(r, \mathrm{Law}(\{\xi(i)\})) \ T^{(\nu)}(x). \tag{5.5}$$

Example 5.1. Let $\mathbf{E}\xi = 0$ and suppose the r.v. $\xi(i)$ satisfies the condition (1.1) uniformly over i. Then for the values $x > e$

$$x^{-r} \ \log^{\gamma}(x) \ L(\log x) \le U(x) \le C_6(r, \gamma, L) \ x^{-r} \ \log^{\gamma+1}(x) \ L(\log x). \tag{5.6}$$

6 Superheavy tails

We will investigate in this section the case when the r.v. ξ has symmetrical distribution such that

$$T_\xi(x) = \frac{K}{\log^\kappa x} L(\log x), \tag{6.1}$$

where as before $K, \kappa = \text{const} > 0, L(z)$ is slowly varying as $z \to \infty$ positive continuous function, $\xi(k), \ k = 1, 2, \ldots$ are independent copies of ξ.

We do not know the exact norming sequence for the sums $\sum_{k=1}^n \xi(k)$. We intend to represent here a more slight result as in the second section.

Let us define for any fixed increasing deterministic tending to $+\infty$ non-random sequence $w(n), \ w(1) = 1$ the following norming sequence B_n :

$$B_n = \exp\left((Kn)^{1/\kappa} \ L^{1/\kappa}\left((Kn)^{1/\kappa} \right) \ w(n) \right) \tag{6.2}$$

and introduce correspondingly

$$S(n) = \frac{\sum_{k=1}^n \xi(k)}{B_n},$$

$$\overline{U}(x) = \sup_n T_{S(n)}(x), \ x > e^2.$$

Theorem 6.1. For some positive finite constant $C = C(\kappa, L, \{w(n)\})$

$$T_\xi(x) \le \overline{U}(x) \le T_\xi(x/C). \tag{6.3a}$$

Further, the sequence $S(n)$ tends in probability to zero as $n \to \infty$:

$$\frac{\sum_{k=1}^n \xi(k)}{B_n} \xrightarrow{\mathbf{P}} 0, \tag{6.3b}$$

a "double Weak" Law of Large Numbers.

Proof. We conclude as before as $t \to 0+$

$$\psi_\xi(t) = 2t \int_0^\infty \sin(tx) \ T_\xi(x) \ dx \sim$$

$$2t \cdot K \int_e^\infty \sin(tx) \ \frac{K}{\log^\kappa x} \ L(\log x) \ dx. \tag{6.4}$$

The exact asymptotic of the last integral as $t \to 0+$ (Fourier transform) is calculated in the classical book of A.Zygmund [48], p. 186-188:

$$\psi_\xi(t) \sim K \ |\log t|^{-\kappa} \ L(|\log t|). \tag{6.5}$$

Calculating the characteristical function for the sequence $S(n)$, we conclude by means of estimate (2.2)

$$\sup_n \left| \psi_{S(n)}(t) \right| \le K_1 \ |\log t|^{-\kappa} \ L(|\log t|), \ t \in (0, 1/e), \tag{6.6}$$

and

$$\lim_{n\to\infty}\left|\psi_{S(n)}(t)\right|\to 0. \tag{6.7}$$

The second proposition (6.3b) of theorem 6.1 follows immediately from (6.6), the first is proved alike the proof of theorem 2.1.

7 Continuity and Stable Limit theorems for heavy tail random fields

1. Continuity.

Let $\eta(v),\ v\in V$ be separable random field (r.f.) (process) defined aside from the probabilistic space Ω on any set V. We suppose that for arbitrary point $v\in V$ the r.v. $\eta(v)$ satisfies the condition (1.1) up to continuous bilateral bounded multiplicative constant $K(v)$:

$$T_{\eta(v)}(x)=K(v)\ x^{-r}\ \log^{\gamma}(x)\ L(\log x), r=\text{const}\in(1,\infty),\ x>e, \tag{7.1}$$

$$\gamma>-1,\ C_1\le K(v)\le C_2,\ C_1,C_2=\text{const}, 0\le C_1\le C_2<\infty.$$

Without loss of generality we can and do assume that for some fixed non-random value $v_0,\ v_0\in V\ K(v_0)=1$.

Let us introduce the following function:

$$\theta(p)=|\eta(v_0)|_p=\nu_{\eta(v_0)}(p),\ 1\le p<r;$$

then

$$\theta(p)\sim(r-p)^{-\gamma-1}\ L(1/(r-p)),\ p\to r-0.$$

From the equality (7.1) follows that

$$\sup_{v\in V}||\eta(v)||G\theta<\infty. \tag{7.2}$$

The so-called *natural distance* $d(v_1,v_2)$ (more exactly, semi-distance: from the equality $d(v_1,v_2)=0$ does not follow $v_1=v_2$) may be defined by the formula

$$d(v_1,v_2)=||\eta(v_1)-\eta(v_2)||G\theta. \tag{7.3}$$

The boundedness of $d(v_1,v_2)$ follows immediately from (7.2).

Remark 7.1. The continuity of the coefficient $K=K(v)$ is understood relative the distance $d=d(v_1,v_2)$.

We denote as usually the metric entropy of the set V in the distance $d(\cdot,\cdot)$ as $H(V,d,\epsilon)$; recall that $H(V,d,\epsilon)$ is the natural logarithm of the minimal

number of $d-$ closed balls with radius ϵ, $\epsilon > 0$ which cover the set V. By definition,

$$N(V,d,\epsilon) = \exp H(V,d,\epsilon).$$

The classical theorem of Hausdorff tell us that $\forall \epsilon > 0 \ N(V,d,\epsilon) < \infty$ iff the set V is precompact set relative the distance d.

Theorem 7.1. If the following integral converges:

$$\int_0^1 N^{1/r}(V,d,\epsilon) \ H^{\gamma/r}(V,d,\epsilon) \ L^{1/r}(H(V,d,\epsilon)) \ d\epsilon < \infty, \tag{7.4}$$

then the trajectories $\eta(t)$ are $d(\cdot,\cdot)$ continuous with probability one:

$$\mathbf{P}(\eta(\cdot) \in C(V,d)) = 1 \tag{7.5}$$

and moreover

$$\mathbf{P}(\sup_{v \in V} |\eta(v)| \ge x) \le C \ \ x^{-r} \ \log^{\gamma}(x) \ L(\log x), \ \ x \ge e. \tag{7.6}$$

Proof. From the definition of the norm $|| \cdot ||G\theta$ follows the inequality

$$\mathbf{P}\left(\left| \frac{\eta(v_1) - \eta(v_2)}{d(v_1,v_2)} \right| \ge x \right) \le C_1 \ x^{-r} \ \log^{\gamma}(x) \ L(\log x), \ \ x > e. \tag{7.7}$$

By definition, we adopt in (7.7) that $0/0 = 0$.

It remains to use the result of the chapter 4, section 4.3 of the monograph [33].

Remark 7.1. The case when

$$\sup_{v \in V} |\eta(v)|_r < \infty, r \ge 1$$

and the distance

$$\rho(v_1,v_2) = |\eta(v_1) - \eta(v_2)|_r$$

was considered by G.Pizier [38]. Indeed, if

$$\int_0^1 N^{1/r}(V,\rho,\epsilon) \ d\epsilon < \infty,$$

then

$$\mathbf{P}(\eta(\cdot) \in C(V,\rho)) = 1$$

and

$$| \sup_{v \in V} |\eta(v)| \ |_r < \infty.$$

Remark 7.1. Another approach via the so-called majorizing measures, see in [12], [44], [45].

2. Stable and Central Limit theorems.

We assume in addition that the random field $\eta(v)$ is symmetrically distributed. Let $\eta_k(v)$ be independent copies of $\eta(v)$. Define as before the norming sequence $b(n)$ as a solution of equation

$$n^{-1} = b^{-r}(n) \ |\log b(n)|^{\gamma} \ L(\log b(n))$$

in the case $r \le 2$ and $b(n) = \sqrt{n}$ when $r > 2$. Put

$$\beta_n(v) = \frac{1}{b(n)} \sum_{k=1}^{n} \eta_k(v). \tag{7.8}$$

The finite-dimensional distributions of the random fields $\beta_n(v)$ converge as $n \to \infty$ to the finite-dimensional distribution of the random field, which we denote as $\beta(v)$. The last r.f. has a stable distribution when $r < 2$ and has a Gaussian centered distribution with at the same covariation function as $\eta(v)$:

$$\mathbf{E}\beta(v_1)\beta(v_2) = \mathbf{E}\eta(v_1)\eta(v_2).$$

More information about stable distributions in the Banach spaces see in the monograph N.N.Vakhania, V.I.Tarieladze and S.A.Chobanan [47], chapter 5.

We will say as ordinary that when the sequence r.f. $\beta_n(\cdot)$ and r.f. $\beta(\cdot)$ are $d-$ continuous with probability one and the distributions in the space $C(V,d)$ of r.f. $\beta_n(\cdot)$ converge weakly as $n \to \infty$ to the distribution of $\beta(\cdot)$, that the random fields $\eta_k(v), \ k = 1, 2, \ldots$ satisfy the Limit Theorem in the space $C(V,d)$.

In the case $r < 2$ we have the Stable Limit Theorem; when $r \ge 2$ one can say as Central Limit Theorem in this space.

The limit theorems in the Banach spaces was initiated by Yu.V.Prokhorov in [40] and was continued in many works, e.g., [2], [8], [10], [32].

The applications of the limit theorems in Banach spaces in the Monte-Carlo method see in [15], [16].

Theorem 7.2. If the following integral is finite:

$$\int_0^1 N^{1/r}(V,d,\epsilon) \ H^{(\gamma+1)/r}(V,d,\epsilon) \ L^{1/r}(H(V,d,\epsilon)) \ d\epsilon < \infty, \tag{7.9}$$

then the random fields $\eta_k(v), \ k = 1, 2, \ldots$ satisfy the Limit Theorem in the space $C(V,d)$.

Moreover,

$$\sup_n \mathbf{P}(\sup_{v \in V} |\beta_n(v)| \ge x) \le C \ \ x^{-r} \ \log^{\gamma+1}(x) \ L(\log x), \ \ x \ge e. \tag{7.10}$$

Proof. As long as the condition (7.10) is more strong as (7.4), we conclude $\mathbf{P}(\beta_n(\cdot) \in C(V,d)) = \mathbf{P}(\beta(\cdot) \in C(V,d)) = 1$.

It remains to prove the *tightness* of the measures μ_n generated by the sequence $\{\beta_n(\cdot)\}$:

$$\mu_n(A) = \mathbf{P}(\beta_n(\cdot) \in A),$$

where A is Borelian set, in the space $C(V,d)$. We obtain using theorem 2.1 that

$$\sup_n \mathbf{P}\left(\left| \frac{\beta_n(v_1) - \beta_n(v_2)}{d(v_1,v_2)} \right| \ge x \right) \le C_2\ x^{-r}\ \log^{\gamma+1}(x)\ L(\log x),\ \ x > e. \quad (7.11)$$

As before, we adopt by definition in (7.11) that $0/0 = 0$.

It remains to use the result of the chapter 4, section 4.3 of the monograph [33].

3. Applications.

A. We return now to the problem computation of (multiple) *parametric* integral of a view:

$$I(v) = \int_D f(v,y)\ \nu(dy),\ \ v \in V, \quad (7.12)$$

where $\nu(\cdot)$ is again probabilistic measure on the set $D:\ \nu(D) = 1$.

Let $\tau(k),\ k = 1,2,\ldots,n$ be as before independent r.v. with distribution $\nu:\ \mathbf{P}(\tau(k) \in A) = \nu(A)$. The Monte-Carlo consistent estimation $I_n(v)$ of an integral $I(t)$ is

$$I_n(v) = n^{-1} \sum_{k=1}^n f(v,\tau(k)). \quad (7.13)$$

Suppose for some $r \in (1,2)$ and for all the values $v \in V$

$$\mathbf{E}|f(v,\tau(1))|^r < \infty$$

or more generally that the r.v. $f(v,\tau(k)) - I(v)$ satisfies the condition (1.1); uniformly in v. In order to construct a non-asymptotical confidence interval for I of a reliability $1-\delta,\ \delta = 0.05; 0.01$ etc. in uniform over $v \in V$ we consider the probability

$$U_n(x) = \mathbf{P}\left(\sup_{v \in V} b(n)^{-1} \left| \sum_{k=1}^n (f(v,\tau(k) - I(v))) \right| > x \right). \quad (7.14)$$

Note that if the sequence of r.f. $\{f(v,\tau(k) - I(v))\}$ satisfy Limit Theorem, then

$$\forall x > 0\ \lim_{n\to\infty} U_n(x) \to U(x), \quad (7.15)$$

where

$$U(x) = \mathbf{P}\left(\sup_{v \in V} |\zeta(v)| > x \right), \quad (7.16)$$

$\zeta(v) = \zeta(\omega,v)$ is stable or Gaussian random field.

The asymptotical or non-asymptotical behavior of $U(x)$ as $x \to \infty$ in both the cases: SLT or CLT is known, see, e.g. [47], chapter 5; [37], [33], chapter 3.

Therefore, we conclude asymptotically as $n \to \infty$ denoting by $X(\delta)$ the solution of an equation

$$U(X(\delta)) = \delta$$

that with probability at least $1 - \delta$ in the uniform norm

$$\sup_{v \in V} |I_n(v) - I(v)| \le X(\delta) b(n)/n. \tag{7.17}$$

Note that as before

$$\lim_{n \to \infty} b(n)/n = 0.$$

Notice that it may be used *non-asymptotical approach,* where the probability $U_n(x)$ allows the evaluating as follows:

$$U_n(x) \le \sup_n U_n(x) \le C \ x^{-r} \ \log^{\gamma+1}(x) \ L(\log x). \tag{7.18}$$

B. Analogous application appears in statistics. Indeed, let us consider the following classical scheme of date-process elaboration.

$$\eta_k(v) = \theta(v) + \xi_k(v), \ k = 1, 2, \ldots, n; \tag{7.19}$$

where $\theta(v), \ v \in V$ is unknown deterministic function, $\{\xi(k)\}$ are i.,i.d. centered r.f. satisfying the condition (1.1) with $r > 1$ (additive noise with heavy tail).

The consistent estimation of the functional parameter $\theta(v)$ has a view

$$\hat{\theta}_n(v) = n^{-1} \sum_{k=1}^{n} \eta_k(v). \tag{7.20}$$

We conclude as before that with probability at least $1 - \delta$ under formulated above conditions and notations

$$\sup_{v \in V} \left| \hat{\theta}_n(v) - \theta(v) \right| \le X(\delta) b(n)/n. \tag{7.21}$$

8 Concluding remarks

A. Non-symmetrical case.

The results of the second and third section remains true still without restriction of symmetrical distribution of the independent r.v. $\{\xi(i)\}$. Indeed, it is sufficient to assume in the case $r \in (1, 2]$ in addition to the equality (1.1)

$$\mathbf{E}\xi(i) = 0$$

and in the case $r = 1$

$$\sup_{a>0} \mathbf{E}|\xi(i)\ I(|\xi(i)| \le a)| < \infty;$$

see, e.g., [1], [5]. The proof may be obtained also from the symmetrization arguments, see also [28].

B. Not identical distributed r.v.

It is not necessary to suppose also in the independent case when $r > 2$ as in the remark 4.2 that the r.v. $\xi(i)$ are identical distributed; it is sufficient to assume in addition that the r.v. $\{\xi(i)\}$ are independent, centered and such that for some positive finite constants $C_1,\ C_2$

$$C_1\ x^{-r}\ \log^{\gamma}(x)\ L(\log x) \le T_{\xi(i)}(x) \le C_2\ x^{-r}\ \log^{\gamma}(x)\ L(\log x),\ \ x > e.$$

C. About calculation of the norming sequence.

We investigate here the equation $n\psi(1/b(n)) = 1$ for the norming sequence $\{b(n)\}$ in the case $r < 2$.

We have under condition (1.1):

$$n^{-1} = b^{-r}(n)\ |\log b(n)|^{\gamma}\ L(\log b(n)). \tag{8.1}$$

It is reasonable to assume that as $n \to \infty$

$$b(n) \sim n^{1/r}\ \log^{\gamma/r}(n)\ L^{1/r}(\log n). \tag{8.2}$$

We obtain substituting into (6.1C) analogously to the classical monograph of E. Seneta [43], p. 29-32 that the asymptotical expression for $b(n)$ is true when the following condition holds:

$$L\left(\frac{X^{1/r}}{\log^{\gamma/r} X \cdot L^{1/r}(\log X)}\right) \asymp L(X),\ \ X \to \infty. \tag{8.3}$$

Note that the condition 6.3C is satisfied for the function, e.g.,

$$L(X) = C\ (\log X)^{\Delta},\ \ \Delta = \mathrm{const}\,.$$

D. Tail comparison through moments inequalities.

If for two r.v. ξ and η $T_\xi(x) \le T_\eta(x),\ x > 0$, then evidently

$$|\xi|_p \le |\eta|_p,\ \ p \ge 0. \tag{8.4}$$

We discuss in this pilcrow the inverse problem. Indeed, suppose the inequality (6.1) holds. Our purpose is to obtain the estimate the upper bound for tail probability $T_\xi(x)$.

In detail, assume that for *any values* p from the non-trivial segment $p \in [1, r),\ r \in (1, \infty)$

$$|\xi|_p \le |\eta|_p, \ p \ge 0. \tag{8.5}$$

It follows from Tchebychev's inequality

$$T_\xi(x) \le x^{-p} \ |\eta|_p^p, \ x > 0, \ p \in \operatorname{supp} \nu_\eta,$$

therefore

$$T_\xi(x) \le \inf_{p \in \operatorname{supp} \nu_\eta} \left[x^{-p} \ |\eta|_p^p \right], \ x > 0. \tag{8.6}$$

If for instance $\operatorname{supp} \nu_\eta = [1, r)$, or equally when $\nu_\eta(r+0) = \infty$ for some $r > 1$, then

$$T_\xi(x) \le \inf_{p \in [1,r)} \left[x^{-p} \ |\eta|_p^p \right], \ x > 0. \tag{8.7}$$

We get, e.g. in particular choosing in (9.7) the value $p = r - C/\log x, \ C =$ const > 0 for sufficiently greatest values x

$$T_\xi(x) \le e^C \ x^{-r} \ \mathbf{E}|\eta|^{r - C/\log x}. \tag{8.8}$$

E. Non-uniform norming sequence.

M.Braverman in the article [5] considered a more general as uniform norming vector $a = \{a(1), a(2), \ldots, a(n)\}$. In detail, let $r \in (0,2)$ and let $\{\xi(k)\}, \ k = 1, 2, \ldots, n$ be again the independent copies of the symmetrical r.v. ξ satisfying the condition (1.1). Put

$$U^{(a)}(x) = \mathbf{P} \left(\left| \sum_{k=1}^n a(k) \xi(k) \right| \right). \tag{8.9}$$

M.Braverman in [5] introduced an Orlicz space of numerical sequences $a = \{a(1), a(2), \ldots, a(n)\}$ by means of Orlicz's function $\psi(t) = \psi_\xi(t)$:

$$||a||_\psi = \inf\{t, t > 0, \sum_{k=1}^n \psi(|a(k)|/t) \le 1\}, \tag{8.10}$$

and proved that the unit ball in this space is natural norming sequence in the sense of $L_p, \ p < r$ boundary.

Denote

$$\overline{U}(x) = \sup_{a: ||a||_\psi \le 1} U^{(a)}(x). \tag{8.11}$$

It may be proved analogously theorem 2.1 that

$$\overline{U}(x) \le 0.5x \int_{-2/x}^{2/x} \overline{\psi}(t) \ dt, \ x > e. \tag{8.12}$$

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